

GROUP ACTIONS ON POSETS

ERIC BABSON AND DMITRY N. KOZLOV

ABSTRACT. In this paper we study quotients of posets by group actions. In order to define the quotient correctly we enlarge the considered class of categories from posets to loopfree categories: categories without nontrivial automorphisms and inverses. We view group actions as certain functors and define the quotients as colimits of these functors. The advantage of this definition over studying the quotient poset (which in our language is the colimit in the poset category) is that the realization of the quotient loopfree category is more often homeomorphic to the quotient of the realization of the original poset. We give conditions under which the quotient commutes with the nerve functor, as well as conditions which guarantee that the quotient is again a poset.

1. INTRODUCTION

Assume that we have a finite group G acting on a poset P in an order-preserving way. The purpose of this chapter is to compare the various constructions of the quotient, associated with this action. Our basic suggestion is to view P as a category and the group action as a functor from G to **Cat**. Then, it is natural to define P/G to be the colimit of this functor. As a result P/G is in general a category, not a poset.

After getting a hand on the formal setting in Section 2 we proceed in Section 3 with imposing different conditions on the group action. We give conditions for each of the following properties to be satisfied:

- (1) the morphisms of P/G are exactly the orbits of the morphisms of P , we call it *regularity*;
- (2) the quotient construction commutes with Quillen's nerve functor;
- (3) P/G is again a poset.

Furthermore, we study the class of categories which can be seen as the "quotient closure" of the set of all finite posets: loopfree categories.

In Section 4 we draw connections to determining the multiplicity of the trivial character in the induced representations of G on the homology groups of the nerve of the category, derive a formula for the Möbius function of P/G and, based on formulae of Sundaram and Welker, [18], give a quotient analog of Goresky-MacPherson formulae.

As another example where these methods proved to be essential we would like to mention the computation of the homology groups of the deleted symmetric join of an infinite simplex, see [1].

Date: February 1, 2008

Mathematical Subject Classification: 05E25, 06A11.

Research at MSRI was supported in part by NSF grant DMS-9022140. The first author was supported by an NSF Postdoctoral Fellowship. The second author was supported by the Swedish Science Council Postdoctoral Fellowship M-PD 11292-303.

2. FORMALIZATION OF GROUP ACTIONS AND THE MAIN QUESTION

2.1. Preliminaries.

For a small category K denote the set of its objects by $\mathcal{O}(K)$ and the set of its morphisms by $\mathcal{M}(K)$. For every $a \in \mathcal{O}(K)$ there is exactly one identity morphism which we denote id_a , this allows us to identify $\mathcal{O}(K)$ with a subset of $\mathcal{M}(K)$. If m is a morphism of K from a to b , we write $m \in \mathcal{M}_K(a, b)$, $\partial^\bullet m = a$ and $\partial_\bullet m = b$. The morphism m has an inverse $m^{-1} \in \mathcal{M}_K(b, a)$, if $m \circ m^{-1} = \text{id}_a$ and $m^{-1} \circ m = \text{id}_b$. If only the identity morphisms have inverses in K then K is said to be a category without inverses.

We denote the category of all small categories by **Cat**. If $K_1, K_2 \in \mathcal{O}(\mathbf{Cat})$ we denote by $\mathcal{F}(K_1, K_2)$ the set of functors from K_1 to K_2 . We need three full subcategories of **Cat**: **P** the category of posets, (which are categories with at most one morphism, denoted $(x \rightarrow y)$, between any two objects x, y), **L** the category of loopfree categories (see Definition 3.9), and **Grp** the category of groups, (which are categories with a single element, morphisms given by the group elements and the law of composition given by group multiplication). Finally, **1** is the terminal object of **Cat**, that is, the category with one element, and one (identity) morphism. The other two categories we use are **Top**, the category of topological spaces, and **SS**, the category of simplicial sets.

We are also interested in the functors $\Delta : \mathbf{Cat} \rightarrow \mathbf{SS}$ and $\mathcal{R} : \mathbf{SS} \rightarrow \mathbf{Top}$. The composition is denoted $\tilde{\Delta} : \mathbf{Cat} \rightarrow \mathbf{Top}$. Here, Δ is the nerve functor, see Appendix B, or [13, 14, 15]. In particular, the simplices of $\Delta(K)$ are chains of morphisms in K , with degenerate simplices corresponding to chains that include identity morphisms, see [5, 19]. \mathcal{R} is the topological realization functor, see [11].

Note that both Δ and $\tilde{\Delta}$ have weak homotopy inverses, i.e., functors $\xi : \mathbf{SS} \rightarrow \mathbf{Cat}$ and $\tilde{\xi} : \mathbf{Top} \rightarrow \mathbf{Cat}$ such that $\tilde{\Delta} \circ \tilde{\xi}$ is homotopic to the identity, and $\mathcal{R} \circ \Delta \circ \xi$ is homotopic to \mathcal{R} , see [4].

We recall here the definition of a colimit (see [10, 12]).

Definition 2.1. Let K_1 and K_2 be categories and $X \in \mathcal{F}(K_1, K_2)$. A *sink* of X is a pair consisting of $L \in \mathcal{O}(K_2)$, and a collection of morphisms $\{\lambda_s \in \mathcal{M}_{K_2}(X(s), L)\}_{s \in \mathcal{O}(K_1)}$, such that if $\alpha \in \mathcal{M}_{K_1}(s_1, s_2)$ then $\lambda_{s_2} \circ X(\alpha) = \lambda_{s_1}$. (One way to think of this collection of morphisms is as a natural transformation between the functors X and $X' = X_1 \circ X_2$, where X_2 is the terminal functor $X_2 : K_1 \rightarrow \mathbf{1}$ and $X_1 : \mathbf{1} \rightarrow K_2$ takes the object of **1** to L). When $(L, \{\lambda_s\})$ is universal with respect to this property we call it the **colimit** of X and write $L = \text{colim } X$.

2.2. Definition of the quotient and formulation of the main problem.

Our main object of study is described in the following definition.

Definition 2.2. We say that a group G **acts on** a category K if there is a functor $\mathcal{A}_K : G \rightarrow \mathbf{Cat}$ which takes the unique object of G to K . The colimit of \mathcal{A}_K is called the **quotient** of K by the action of G and is denoted by K/G .

To simplify notations, we identify $\mathcal{A}_K g$ with g itself. Furthermore, in Definition 2.2 the category **Cat** can be replaced with any category C , then $K, K/G \in \mathcal{O}(C)$. Important special case is $C = \mathbf{SS}$. It arises when $K \in \mathcal{O}(\mathbf{Cat})$ and we consider $\text{colim } \Delta \circ \mathcal{A}_K = \Delta(K)/G$.

Main Problem. Understand the relation between the topological and the categorical quotients, that is, between $\Delta(K/G)$ and $\Delta(K)/G$.

To start with, by the universal property of colimits there exists a canonical surjection $\lambda : \Delta(K)/G \rightarrow \Delta(K/G)$. In the next section we give combinatorial conditions under which this map is an isomorphism.

The general theory tells us that if G acts on the category K , then the colimit K/G exists, since **Cat** is cocomplete. We shall now give an explicit description.

An explicit description of the category K/G .

When x is a morphism of K , denote by Gx the orbit of x under the action of G . We have $\mathcal{O}(K/G) = \{Ga \mid a \in \mathcal{O}(K)\}$. The situation with morphisms is more complicated. Define a relation \leftrightarrow on the set $\mathcal{M}(K)$ by setting $x \leftrightarrow y$, iff there are decompositions $x = x_1 \circ \cdots \circ x_t$ and $y = y_1 \circ \cdots \circ y_t$ with $Gy_i = Gx_i$ for all $i \in [t]$. The relation \leftrightarrow is reflexive and symmetric since G has identity and inverses, however it is not in general transitive. Let \sim be the transitive closure of \leftrightarrow , it is clearly an equivalence relation. Denote the \sim equivalence class of x by $[x]$. Note that \sim is the minimal equivalence relation on $\mathcal{M}(K)$ closed under the G action and under composition; that is, with $a \sim ga$ for any $g \in G$, and if $x \sim x'$ and $y \sim y'$ and $x \circ x'$ and $y \circ y'$ are defined then $x \circ x' \sim y \circ y'$. It is not difficult to check that the set $\{[x] \mid x \in \mathcal{M}(K)\}$ with the relations $\partial_\bullet[x] = [\partial_\bullet x]$, $\partial^\bullet[x] = [\partial^\bullet x]$ and $[x] \circ [y] = [x \circ y]$ (whenever the composition $x \circ y$ is defined), are the morphisms of the category K/G .

Note that if P is a poset with a G action, the quotient taken in **Cat** need not be a poset, and hence may differ from the poset quotient.

Example 2.3. Let P be the center poset in the figure below. Let S_2 act on P by simultaneously permuting a with b and c with d . (I) shows P/S_2 in **P** and (II) shows P/S_2 in **Cat**. Note that in this case the quotient in **Cat** commutes with the functor Δ (the canonical surjection λ is an isomorphism), whereas the quotient in **P** does not.

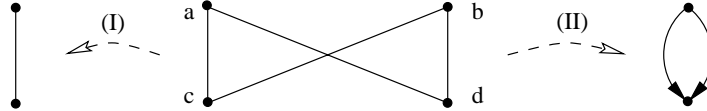


Figure 5.1

3. CONDITIONS ON GROUP ACTIONS

3.1. Outline of the results and surjectiveness of the canonical map.

In this section we consider combinatorial conditions for a group G acting on a category K which ensure that the quotient by the group action commutes with the nerve functor. If $\mathcal{A}_K : G \rightarrow \mathbf{Cat}$ is a group action on a category K then $\Delta \circ \mathcal{A}_K : G \rightarrow \mathbf{SS}$ is the associated group action on the nerve of K . It is clear that $\Delta(K/G)$ is a sink for $\Delta \circ \mathcal{A}_K$, and hence, as previously mentioned, the universal property of colimits gives a canonical map $\lambda : \Delta(K)/G \rightarrow \Delta(K/G)$. We wish to find conditions under which λ is an isomorphism.

First we prove in Proposition 3.1 that λ is always surjective. Furthermore, $Ga = [a]$ for $a \in \mathcal{O}(K)$, which means that, restricted to 0-skeleta, λ is an isomorphism. If the two simplicial spaces were simplicial complexes (only one face for any fixed vertex set), this would suffice to show isomorphism. Neither one is a simplicial complex in general, but while the quotient of a complex $\Delta(K)/G$ can

have simplices with fairly arbitrary face sets in common, $\Delta(K/G)$ has only one face for any fixed edge set, since it is a nerve of a category. Thus for λ to be an isomorphism it is necessary and sufficient to find conditions under which

- 1) λ is an isomorphism restricted to 1-skeleta;
- 2) $\Delta(K)/G$ has only one face with any given set of edges.

We will give conditions equivalent to λ being an isomorphism, and then give some stronger conditions that are often easier to check, the strongest of which is also inherited by the action of any subgroup H of G acting on K .

First note that a simplex of $\Delta(K/G)$ is a sequence $([m_1], \dots, [m_t])$, $m_i \in \mathcal{M}(K)$, with $\partial_\bullet[m_{i-1}] = \partial^\bullet[m_i]$, which we will call a *chain*. On the other hand a simplex of $\Delta(K)/G$ is an orbit of a sequence (n_1, \dots, n_t) , $n_i \in \mathcal{M}(K)$, with $\partial_\bullet n_{i-1} = \partial^\bullet n_i$, which we denote $G(n_1, \dots, n_t)$. The canonical map λ is given by $\lambda(G(n_1, \dots, n_t)) = ([n_1], \dots, [n_t])$.

Proposition 3.1. *Let K be a category and G a group acting on K . The canonical map $\lambda : \Delta(K)/G \rightarrow \Delta(K/G)$ is surjective.*

Proof. By the above description of λ it suffices to fix a chain $([m_1], \dots, [m_t])$ and find a chain (n_1, \dots, n_t) with $[n_i] = [m_i]$. The proof is by induction on t . The case $t = 1$ is obvious, just take $n_1 = m_1$.

Assume now that we have found n_1, \dots, n_{t-1} , so that $[n_i] = [m_i]$, for $i = 1, \dots, t-1$, and n_1, \dots, n_{t-1} compose, i.e., $\partial^\bullet n_i = \partial_\bullet n_{i+1}$, for $i = 1, \dots, t-2$. Since $[\partial_\bullet n_{t-1}] = [\partial_\bullet m_{t-1}] = [\partial^\bullet m_t]$, we can find $g \in G$, such that $g\partial^\bullet m_t = \partial_\bullet n_{t-1}$. If we now take $n_t = gm_t$, we see that n_{t-1} and n_t compose, and $[n_t] = [m_t]$, which provides a proof for the induction step. \square

3.2. Conditions for injectiveness of the canonical projection.

Definition 3.2. *Let K be a category and G a group acting on K . We say that this action satisfies **Condition (R)** if the following is true: If $x, y_a, y_b \in \mathcal{M}(K)$, $\partial_\bullet x = \partial^\bullet y_a = \partial^\bullet y_b$ and $Gy_a = Gy_b$, then $G(x \circ y_a) = G(x \circ y_b)$.*

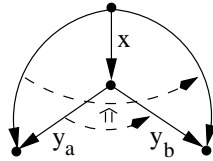


Figure 5.2

We say in such case that G acts *regularly* on K .

Proposition 3.3. *Let K be a category and G a group acting on K . This action satisfies Condition (R) iff the canonical surjection $\lambda : \Delta(K)/G \rightarrow \Delta(K/G)$ is injective on 1-skeleta.*

Proof. The injectiveness of λ on 1-skeleta is equivalent to requiring that $Gm = [m]$, for all $m \in \mathcal{M}(K)$, while Condition (R) is equivalent to requiring that $G(m \circ Gn) = G(m \circ n)$, for all $m, n \in \mathcal{M}(K)$ with $\partial_\bullet m = \partial^\bullet n$; here $m \circ Gn$ means the set of all $m \circ gn$ for which the composition is defined.

Assume that λ is injective on 1-skeleta. Then we have the following computation:

$$G(m \circ Gn) = Gm \circ Gn = [m] \circ [n] = [m \circ n] = G(m \circ n),$$

hence the Condition (R) is satisfied.

Reversely, assume that the Condition (R) is satisfied, that is $G(m \circ Gn) = G(m \circ n)$. Since the equivalence class $[m]$ is generated by G and composition, it suffices to show that orbits are preserved by composition, which is precisely $G(m \circ Gn) = G(m \circ n)$. \square

The following theorem is the main result of this chapter. It provides us with combinatorial conditions which are equivalent to λ being an isomorphism.

Theorem 3.4. *Let K be a category and G a group acting on K . The following two assertions are equivalent for any $t \geq 2$:*

- (1_t) **Condition (C_t).** *If $m_1, \dots, m_{t-1}, m_a, m_b \in \mathcal{M}(K)$ with $\partial^\bullet m_i = \partial_\bullet m_{i-1}$ for all $2 \leq i \leq t-1$, $\partial^\bullet m_a = \partial^\bullet m_b = \partial_\bullet m_{t-1}$, and $Gm_a = Gm_b$, then there is some $g \in G$ such that $gm_a = m_b$ and $gm_i = m_i$ for $1 \leq i \leq t-1$.*
- (2_t) *The canonical surjection $\lambda : \Delta(K)/G \rightarrow \Delta(K/G)$ is injective on t -skeleta.*

In particular, λ is an isomorphism iff (C_t) is satisfied for all $t \geq 2$. If this is the case, we say that Condition (C) is satisfied

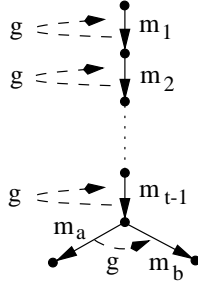


Figure 5.3

Proof. (1_t) is equivalent to $G(m_1, \dots, m_t) = G(m_1, \dots, m_{t-1}, Gm_t)$; this notation is used, as before, for all sequences $(m_1, \dots, m_{t-1}, gm_t)$ which are chains, that is for which $m_1 \circ \dots \circ m_{t-1} \circ gm_t$ is defined. (2_t) implies Condition (R) above, and so can be restated as $G(m_1, \dots, m_t) = (Gm_1, \dots, Gm_t)$.

$$\begin{aligned} \underline{(2_t) \Rightarrow (1_t)} : G(m_1, \dots, m_t) &= (Gm_1, \dots, Gm_t) = G(Gm_1, \dots, Gm_t) \\ &\supseteq G(m_1, \dots, m_{t-1}, Gm_t) \supseteq G(m_1, \dots, m_t). \end{aligned}$$

$$\begin{aligned} \underline{(1_2) \Rightarrow (2_2)} : G(m_1, m_2) &= G(m_1, Gm_2) \\ &= \{g_1(m_1, g_2 m_2) \mid \partial_\bullet m_1 = \partial^\bullet g_2 m_2\} \\ &= \{(g_1 m_1, g_2 m_2) \mid \partial_\bullet g_1 m_1 = \partial^\bullet g_2 m_2\} = (Gm_1, Gm_2). \end{aligned}$$

(1_t) \Rightarrow (2_t), $t \geq 3$: We use induction on t .

$$\begin{aligned} G(m_1, \dots, m_t) &= G(m_1, \dots, m_{t-1}, Gm_t) \\ &= \{(gm_1, \dots, gm_{t-1}, \tilde{g}m_t) \mid \partial_\bullet gm_{t-1} = \partial^\bullet \tilde{g}m_t\} \\ &= \{(g_1 m_1, \dots, g_t m_t) \mid \partial_\bullet g_i m_i = \partial^\bullet g_{i+1} m_{i+1}, i \in [t-1]\} \\ &= (Gm_1, \dots, Gm_t). \quad \square \end{aligned}$$

Example 3.5. A group action which satisfies Condition (C_t) , but does not satisfy Condition (C_{t+1}) . Let P_{t+1} be the order sum of $t + 1$ copies of the 2-element antichain. The automorphism group of P_{t+1} is the direct product of $t + 1$ copies of \mathbb{Z}_2 . Take G to be the index 2 subgroup consisting of elements with an even number of nonidentity terms in the product.

The following condition implies Condition (C) , and is often easier to check.

Condition (S). There exists a set $\{S_m\}_{m \in \mathcal{M}(K)}$, $S_m \subseteq \text{Stab}(m)$, such that

- (1) $S_m \subseteq S_{\partial \bullet m} \subseteq S_{m'}$, for any $m' \in \mathcal{M}(K)$, such that $\partial \bullet m' = \partial \bullet m$;
- (2) $S_{\partial \bullet m}$ acts transitively on $\{gm \mid g \in \text{Stab}(\partial \bullet m)\}$, for any $m \in \mathcal{M}(K)$.

Proposition 3.6. Condition (S) implies Condition (C).

Proof. Let $m_1, \dots, m_{t-1}, m_a, m_b$ and g be as in Condition (C), then, since $g \in \text{Stab}(\partial \bullet m_a)$, there must exist $\tilde{g} \in S_{\partial \bullet m_a}$ such that $\tilde{g}(m_a) = m_b$. From (1) above one can conclude that $\tilde{g}(m_i) = m_i$, for $i \in [t - 1]$. \square

We say that the **strong** Condition (S) is satisfied if Condition (S) is satisfied with $S_a = \text{Stab}(a)$. Clearly, in such a case part (2) of the Condition (S) is obsolete.

Example 3.7. A group action satisfying Condition (S), but not the strong Condition (S). Let $K = \mathcal{B}_n$, lattice of all subsets of $[n]$ ordered by inclusion, and let $G = \mathcal{S}_n$ act on \mathcal{B}_n by permuting the ground set $[n]$. Clearly, for $A \subseteq [n]$, we have $\text{Stab}(A) = \mathcal{S}_A \times \mathcal{S}_{[n] \setminus A}$, where, for $X \subseteq [n]$, \mathcal{S}_X denotes the subgroup of \mathcal{S}_n which fixes elements of $[n] \setminus X$ and acts as a permutation group on the set X . Since $A > B$ means $A \supset B$, condition (1) of (S) is not satisfied for $S_A = \text{Stab}(A)$: $\mathcal{S}_A \times \mathcal{S}_{[n] \setminus A} \not\supseteq \mathcal{S}_B \times \mathcal{S}_{[n] \setminus B}$. However, we can set $S_A = \mathcal{S}_A$. It is easy to check that for this choice of $\{S_A\}_{A \in \mathcal{B}_n}$ Condition (S) is satisfied.

We close the discussion of the conditions stated above by the following proposition.

Proposition 3.8.

- 1) The sets of group actions which satisfy Condition (C) or Condition (S) are closed under taking the restriction of the group action to a subcategory.
- 2) Assume a finite group G acts on a poset P , so that Condition (S) is satisfied. Let $x \in P$ and $S_x \subseteq H \subseteq \text{Stab}(x)$, then Condition (S) is satisfied for the action of H on $P_{\leq x}$.
- 3) Assume a finite group G acts on a category K , so that Condition (S) is satisfied with $S_a = \text{Stab}(a)$ (strong version), and H is a subgroup of G . Then the strong version of Condition (S) is again satisfied for the action of H on K .

Proof. 1) and 3) are obvious. To show 2) observe that for $a \leq x$ we have $S_a \subseteq S_x \subseteq H$, hence $S_a \subseteq H \cap \text{Stab}(a)$. Thus condition (1) remains true. Condition (2) is true since $\{g(b) \mid g \in \text{Stab}(a)\} \supseteq \{g(b) \mid g \in \text{Stab}(a) \cap H\}$. \square

3.3. Conditions for the categories to be closed under taking quotients.

Next, we are concerned with finding out what categories one may get as a quotient of a poset by a group action. In particular, we ask: *in which cases is the quotient again a poset?* To answer that question, it is convenient to use the following class of categories.

Definition 3.9. A category is called **loopfree** if it has no inverses and no non-identity automorphisms.

Intuitively, one may think of loopfree categories as those which can be drawn so that all nontrivial morphisms point down. To familiarize us with the notion of a loopfree category we make the following observations:

- K is loopfree iff for any $x, y \in \mathcal{O}(K)$, $x \neq y$, only one of the sets $\mathcal{M}_K(x, y)$ and $\mathcal{M}_K(y, x)$ is non-empty and $\mathcal{M}_K(x, x) = \{\text{id}_x\}$;
- a poset is a loopfree category;
- a barycentric subdivision of an arbitrary category is a loopfree category;
- a barycentric subdivision of a loopfree category is a poset;
- if K is a loopfree category, then there exists a partial order \geq on the set $\mathcal{O}(K)$ such that $\mathcal{M}_K(x, y) \neq \emptyset$ implies $x \geq y$.

Definition 3.10. Suppose K is a small category, and $T \in \mathcal{F}(K, K)$. We say that T is **horizontal** if for any $x \in \mathcal{O}(K)$, if $T(x) \neq x$, then $\mathcal{M}_K(x, T(x)) = \mathcal{M}_K(T(x), x) = \emptyset$. When a group G acts on K , we say that the action is horizontal if each $g \in G$ is a horizontal functor.

When K is a finite loopfree category, the action is always horizontal. Another example of horizontal actions is given by rank preserving action on a (not necessarily finite) poset. We have the following useful property:

Proposition 3.11. Let P be a finite loopfree category and $T \in \mathcal{F}(P, P)$ be a horizontal functor. Let $\tilde{T} \in \mathcal{F}(\Delta(P), \Delta(P))$ be the induced functor, i.e., $\tilde{T} = \Delta(T)$. Then $\Delta(P_T) = \Delta(P)_{\tilde{T}}$, where P_T denotes the subcategory of P fixed by T and $\Delta(P)_{\tilde{T}}$ denotes the subcomplex of $\Delta(P)$ fixed by \tilde{T} .

Proof. Obviously, $\Delta(P_T) \subseteq \Delta(P)_{\tilde{T}}$. On the other hand, if for some $x \in \Delta(P)$ we have $\tilde{T}(x) = x$, then the minimal simplex σ , which contains x , is fixed as a set and, since the order of simplices is preserved by T , σ is fixed by T pointwise, thus $x \in \Delta(P_T)$. \square

The class of loopfree categories can be seen as the closure of the class of posets under the operation of taking the quotient by a horizontal group action. More precisely, we have:

Proposition 3.12. The quotient of a loopfree category by a horizontal action is again a loopfree category. In particular, the quotient of a poset by a horizontal action is a loopfree category.

Proof. Let K be a loopfree category and assume G acts on K horizontally. First observe that $\mathcal{M}_{K/G}([x]) = \{\text{id}_{[x]}\}$. Because if $m \in \mathcal{M}_{K/G}([x])$, then there exist $x_1, x_2 \in \mathcal{O}(K)$, $\tilde{m} \in \mathcal{M}_K(x_1, x_2)$, such that $[x_1] = [x_2]$, $[\tilde{m}] = m$. Then $gx_1 = x_2$ for some $g \in G$, hence, since g is a horizontal functor, $x_1 = x_2$ and since K is loopfree we get $\tilde{m} = \text{id}_{x_1}$.

Let us show that for $[x] \neq [y]$ at most one of the sets $\mathcal{M}_{K/G}([x], [y])$ and $\mathcal{M}_{K/G}([y], [x])$ is nonempty. Assume the contrary and pick $m_1 \in \mathcal{M}_{K/G}([x], [y])$, $m_2 \in \mathcal{M}_{K/G}([y], [x])$. Then there exist $x_1, x_2, y_1, y_2 \in \mathcal{O}(K)$, $\tilde{m}_1 \in \mathcal{M}_K(x_1, y_1)$, $\tilde{m}_2 \in \mathcal{M}_K(y_2, x_2)$ such that $[x_1] = [x_2] = [x]$, $[y_1] = [y_2] = [y]$, $[\tilde{m}_1] = [m_1]$, $[\tilde{m}_2] = [m_2]$. Choose $g \in G$ such that $gy_1 = y_2$. Then $[gx_1] = [x_2] = [x]$ and we have $g\tilde{m}_1 \in \mathcal{M}_K(gx_1, y_2)$, so $\tilde{m}_2 \circ g\tilde{m}_1 \in \mathcal{M}_K(gx_1, x_2)$. Since K is loopfree we conclude that $gx_1 = x_2$, but then both $\mathcal{M}_K(x_2, y_2)$ and $\mathcal{M}_K(y_2, x_2)$ are nonempty, which contradicts to the fact that K is loopfree. \square

Next, we shall state a condition under which the quotient of a loopfree category is a poset.

Proposition 3.13. *Let K be a loopfree category and let G act on K . The following two assertions are equivalent:*

- (1) **Condition (SR).** *If $x, y \in \mathcal{M}(K)$, $\partial^\bullet x = \partial^\bullet y$ and $G\partial_\bullet x = G\partial_\bullet y$, then $Gx = Gy$.*
- (2) *G acts regularly on K and K/G is a poset.*

Proof. (2) \Rightarrow (1). Follows immediately from the regularity of the action of G and the fact that there must be only one morphism between $[\partial^\bullet x](= [\partial^\bullet y])$ and $[\partial_\bullet x](= [\partial_\bullet y])$.

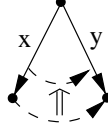


Figure 5.4

(1) \Rightarrow (2). Obviously (SR) \Rightarrow (R), hence the action of G is regular. Furthermore, if $x, y \in \mathcal{M}(K)$ and there exist $g_1, g_2 \in G$ such that $g_1 \partial^\bullet x = \partial^\bullet y$ and $g_2 \partial_\bullet x = \partial_\bullet y$, then we can replace x by $g_1 x$ and reduce the situation to the one described in Condition (SR), namely that $\partial^\bullet x = \partial^\bullet y$. Applying Condition (SR) and acting with g_1^{-1} yields the result. \square

When K is a poset, Condition (SR) can be stated in simpler terms.

Condition (SRP). If $a, b, c \in K$, such that $a \geq b$, $a \geq c$ and there exists $g \in G$ such that $g(b) = c$, then there exists $\tilde{g} \in G$ such that $\tilde{g}(a) = a$ and $\tilde{g}(b) = c$.

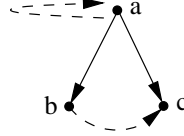


Figure 5.5

That is, for any $a, b \in P$, such that $a \geq b$, we require that the stabilizer of a acts transitively on Gb .

Proposition 3.14. *Let P be a poset and assume G acts on P . The action of G on P induces an action on the barycentric subdivision BdP (the poset of all chains of P ordered by inclusion). This action satisfies Condition (S), hence it is regular and $\Delta(BdP)/G \cong \Delta((BdP)/G)$. Moreover, if the action of G on P is horizontal, then $(BdP)/G$ is a poset.*

Proof. Let us choose chains b, c and $a = (a_1 > \dots > a_t)$, such that $a \geq b$ and $a \geq c$. Then $b = (a_{i_1} > \dots > a_{i_l})$, $c = (a_{j_1} > \dots > a_{j_l})$. Assume also that there exists $g \in G$ such that $g(a_{i_s}) = a_{j_s}$ for $s \in [l]$. If g fixes a then it fixes every a_i , $i \in [t]$, hence $b = c$ and Condition (S) follows.

If, moreover, the action of G is horizontal, then again $a_{i_s} = a_{j_s}$, for $s \in [l]$, hence $b = c$ and Condition (SRP) follows. \square

4. APPLICATIONS.

Let us first state two simple, but nevertheless fundamental, facts.

Proposition 4.1. *Assume that G is a finite group which acts on K , a category without inverses, so that condition (C) is satisfied. Then $\beta_i(\Delta(K/G)) = \langle \gamma_i, 1 \rangle$, where γ_i is the induced representation of G on $H_i(\Delta(K))$.*

Proof. $\langle \gamma_i, 1 \rangle = \beta_i(\Delta(K)/G) = \beta_i(\Delta(K/G))$, where the first equality follows from [3, Theorem 1] and the second from Theorem 3.4. \square

Proposition 4.2. *Let K be a finite loopfree category and G a finite group which acts on K , then*

$$\chi(\Delta(K)/G) = \frac{1}{|G|} \sum_{g \in G} \chi(\Delta(K_g)),$$

where K_g denotes the subcategory of K which is fixed by g .

Proof.

$$\chi(\Delta(K)/G) = \frac{1}{|G|} \sum_{g \in G} \chi(\Delta(K)_g) = \frac{1}{|G|} \sum_{g \in G} \chi(\Delta(K_g)),$$

where the first equality follows from [3, Theorem 2] and the second from Proposition 3.11. \square

Proposition 4.2 can be nicely restated in combinatorial language. To do this we need the following definition.

Definition 4.3. *Let K be a category, such that $\Delta(K)$ has finitely many simplices. We define $\mu(K) \stackrel{\text{def}}{=} \tilde{\chi}(\Delta(K))$. We call $\mu(K)$ the **Möbius function** of K .*

Clearly this definition generalizes the Möbius function of a poset. Similar definitions have been given: most notably (and apparently independently) in [6] and [2]. We would like to mention that if K is a finite loopfree category then one has a generalization of the recursive formula for the computation of the Möbius function (which is often taken as a definition of the Möbius function of a poset):

$$\mu(\hat{0}, x) = - \sum_{y \in \mathcal{O}(K), y < x} m_{x,y} \mu(\hat{0}, y),$$

where $m_{x,y} = |\mathcal{M}_K(x, y)|$, $\mu(\hat{0}, \hat{0}) = 1$ and $\mu(\hat{0}, \hat{1}) = \mu(K)$. Here $\hat{0}$ and $\hat{1}$ are adjoint terminal and initial objects.

Proposition 4.4. *Let K be a finite loopfree category, G a finite group acting on K , such that condition (C) is satisfied. Then*

$$\mu(K/G) = \frac{1}{|G|} \sum_{g \in G} \mu(K_g).$$

Proof. Follows from Propositions 4.2 and 3.4 and the definition of the Möbius function. \square

As another application we obtain a quotient analog of the Goresky-MacPherson formulae.

Proposition 4.5. *Let \mathcal{A} be a subspace arrangement in \mathbb{C}^n and let G be a finite group which acts on \mathcal{A} . Assume that the induced action of G on $\mathcal{L}_{\mathcal{A}}$ (the intersection lattice of \mathcal{A}) satisfies condition (C). Then*

$$\beta^{n-1-i}(\mathcal{M}_{\mathcal{A}}/G) = \beta_i(\mathcal{L}_{\mathcal{A}}/G) = \sum_{x \in \mathcal{L}_{\mathcal{A}}^{>\hat{0}}/G} \beta_{i-\dim x-1}(\Delta((\hat{0}, x)/\text{Stab}(x))).$$

Proof. Follows from [18, Corollaries 2.8 and 2.10], Theorem 3.4 and Proposition 3.8. \square

Remark 4.6. *A similar (though less pretty) formula can be derived for real subspace arrangements, cf. [18, Theorems 2.4 and 2.5].*

Acknowledgments. We would like to thank Eva-Maria Feichtner for the careful reading of this paper.

REFERENCES

- [1] E. Babson, D.N. Kozlov, *Diagrams of classifying spaces and the symmetric deleted join*, preprint 1998.
- [2] M. Content, F. Lemay, P. Leroux, *Catégories de Möbius et fonctorialités: un cadre général pour l'inversion de Möbius*, J. Combin. Theory Ser. A **28** (1980), no. 2, pp. 169–190.
- [3] P.E. Conner, *Concerning the action of a finite group*, Proc. Nat. Acad. Sci. U.S.A. **142**, (1956), 349–351.
- [4] R. Fritsch, D.M. Latch, *Homotopy inverses for nerve*, Math. Z. **177** (1981), 147–179.
- [5] S. Gelfand, Y. Manin, *Methods of homological algebra*, Translated from the 1988 Russian original, Springer-Verlag, Berlin, 1996.
- [6] J. Haigh, *On the Möbius algebra and the Grothendieck ring of a finite category*, J. London Math. Soc. (2) **21** (1980), no. 1, pp. 81–92.
- [7] P. Hanlon, *The fixed-point partition lattices*, Pacific J. Math. **96** (1981), no. 2, pp. 319–341.
- [8] P. Hanlon, *A proof of a conjecture of Stanley concerning partitions of a set*, European J. Combin. **4** (1983), no. 2, 137–141.
- [9] D.N. Kozlov, *Collapsibility of $\Delta(\Pi_n)/S_n$ and some related CW complexes*, Proc. Amer. Math. Soc. **128** (2000), no. 8, 2253–2259.
- [10] S. Mac Lane, *Categories for the working mathematician*, Second edition, Graduate Texts in Mathematics **5**, Springer-Verlag, New York, 1998.
- [11] J. Milnor, *The geometric realization of semi-simplicial complex*, Ann. of Math. **65** (1957), 357–362.
- [12] B. Mitchell, *Theory of categories*, Pure and Applied Mathematics, Vol. XVII, Academic Press, New York-London, 1965.
- [13] D. Quillen, *Higher algebraic K-theory I*, Lecture Notes in Mathematics **341**, Springer-Verlag, Berlin, 1973, pp. 85–148.
- [14] D. Quillen, *Homotopy properties of the poset of nontrivial p -subgroups of a group*, Adv. Math. **28** (1978), no. 2, 101–128.
- [15] G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. No. **34** (1968), 105–112.
- [16] R.P. Stanley, *Some aspects of groups acting on finite posets*, J. Combin. Theory Ser. A **32** (1982), no. 2, 132–161.
- [17] S. Sundaram, *The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice*, Adv. Math. **104** (1994), no. 2, 225–296.
- [18] S. Sundaram, V. Welker, *Group actions on arrangements of linear subspaces and applications to configuration spaces*, Trans. Amer. Math. Soc. **349** (1997), no. 4, 1389–1420.
- [19] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, Cambridge, 1994.

DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, STOCKHOLM, S-100 44,
SWEDEN.

E-mail address: babson@math.washington.edu, kozlov@math.kth.se